# KÄHLER AND PARA-KÄHLER STRUCTURES FOR INFORMATION GEOMETRY

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Let $\mathcal{M}$ be a smooth (real) manifold of even dimensions and $\nabla$ be a (not necessarily torsion-free) connection on it. We study the interaction of $\nabla$ with three compatible geometric structures on $\mathcal{M}$: a pseudo-Riemannian metric $g$, a nondegenerate two-form $\omega$, and a tangent bundle isomorphism $L : T\mathcal{M} \to T\mathcal{M}$. Two special cases of $L$ are: almost complex structure $L^2 = -id$ and almost para-complex structure $L^2 = id$, which will be treated in a unified fashion. When both $g$ and $\omega$ are parallel under a torsion-free $\nabla$, it is well known that $L$ and $\omega$ are integrable, turning an almost (para-)Hermitian manifold $(\mathcal{M}, g, L)$ into an (para-)Kähler manifold $(\mathcal{M}, g, \omega, L)$, where $(g, \omega, L)$ forms "compatible triple". We relax the condition of parallelism under $\nabla$ to the condition of Codazzi coupling with $\nabla$, for each member of the triple.

To this end, we define an *almost Codazzi-(para-)Kähler manifold* $(\mathcal{M}, g, L, \nabla)$ to be an almost (para-)Hermitian manifold $(\mathcal{M}, g, L)$ with an affine connection $\nabla$ (not necessarily torsion-free) which is Codazzi-coupled to both $g$ and $L$. We prove that if $\nabla$ is torsion-free, then $L$ is automatically integrable and $\omega$ is parallel. In this case, $(\mathcal{M}, g, L, \nabla)$ is said to be a *Codazzi-(para-)Kähler manifold*.

**Definitions.** Let $\nabla$ be a torsion-free connection on $\mathcal{M}$, $g$ and $\omega$ be symmetric and skew-symmetric non-degenerate (0,2)-tensor fields respectively, and $L$ be an almost (para-)complex structure. Consider the following relations (for arbitrary vector fields $X, Y, Z$ on $\mathcal{M}$):

(i) $\omega(X, Y) = g(LX, Y)$;

(ii) $g(LX, Y) + g(X, LY) = 0$;

(iii) $\omega(LX, Y) = \omega(LY, X)$;

(iv) $(\nabla_X L)Y = (\nabla_Y L)X$;

(v) $(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$;

(vi) $(\nabla_X \omega)(Y, Z) = 0$.

Conditions (i)-(iii) define a *compatible triple* $(g, \omega, L)$ – any two of the three specifies the third. Condition (iv), (v), and (vi) defines *Codazzi coupling* with $\nabla$ for $L$, $g$, and $\omega$, respectively. We call $(g, \omega, L, \nabla)$ a *compatible quadruple* on $\mathcal{M}$ if Conditions (i)-(vi) are all satisfied.

Our results are shown as the following two main Theorems.

**Theorem 1.** Let $\mathcal{M}$ admit a torsion-free connection $\nabla$, along with any two of the three tensor fields: $g, \omega, L$. Then $\mathcal{M}$ is a Codazzi-(para-)Kähler manifold if and only if *any* of the following conditions holds (which then implies the rest):

1. $(g, L, \nabla)$ satisfy (ii), (iv) and (v);
2. $(\omega, L, \nabla)$ satisfy (iii), (iv) and (vi);
3. $(g, \omega, \nabla)$ satisfy (v) and (vi), in which case $L$ is determined by (i).

Furthermore, $(g, \omega, L, \nabla)$ forms a compatible quadruple on $\mathcal{M}$.

An alternative characterization of the above finding is through relationships among the three transformations of a (not necessarily torsion-free) connection $\nabla$: its $g$-conjugate $\nabla^*$, its $\omega$-conjugate $\nabla^\dagger$, and its $L$-gauge transform $\nabla^L$.

**Theorem 2.** Let $(g, \omega, L)$ be a compatible triple. Then $(id, *, \dagger, L)$ act as the 4-element Klein group on the space of affine connections:

$$\begin{aligned}
&(\nabla^*)^* = (\nabla^\dagger)^\dagger = (\nabla^L)^L = \nabla;\\
&\nabla^* = (\nabla^\dagger)^L = (\nabla^L)^\dagger;\\
&\nabla^\dagger = (\nabla^*)^L = (\nabla^L)^*;\\
&\nabla^L = (\nabla^*)^\dagger = (\nabla^\dagger)^*.
\end{aligned}$$

It follows that any Codazzi-(para-)Kähler manifold admits a Codazzi dual $\nabla^C$ of $\nabla$, defined as $\nabla^* = \nabla^L$, satisfying

(iv) $(\nabla^C_X L)Y = (\nabla^C_Y L)X$;

(v) $(\nabla^C_X g)(Y, Z) = (\nabla^C_Y g)(X, Z)$;

(vi) $(\nabla^C_X \omega)(Y, Z) = 0$.

To summarize: Codazzi-(para-)Kähler manifold is a (para-)Kähler manifold that is simultaneously a statistical manifold. A statistical structure $(g, \nabla)$ can be enhanced to a Codazzi-(para-)Kähler structure, which is a special kind of (para-)Kähler manifold, with the introduction of a "nice enough" $L$ in the sense that $L$ is compatible with $g$ and Codazzi coupled to $\nabla$. When $\nabla$ is dually flat (i.e., Hessian statistical structure), we get the so-called "special Kähler geometry."
**Keywords:** symplectic, Codazzi dual, compatible triple, compatible quadruple

# References

[1] T. Fei and J. Zhang (preprint). Interaction of Codazzi coupling and (para)-Kähler geometry.