Information geometry induced from sandwiched Rényi α-divergence

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Recently, Müller-Lennert *et al.* [9] and Wilde *et al.* [11] independently proposed an extension of the Rényi relative entropy [10] to the quantum domain. Let $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}_{\rm sa}(\mathcal{H})$ denote the set of linear operators and selfadjoint operators on a finite dimensional complex Hilbert space $\mathcal{H}$, and let $\mathcal{L}_{+}(\mathcal{H})$ and $\mathcal{L}_{++}(\mathcal{H})$ denote the subset of $\mathcal{L}_{\rm sa}(\mathcal{H})$ comprising positive operators and strictly positive operators. Given $\rho, \sigma \in \mathcal{L}_{+}(\mathcal{H})$ with $\rho \neq 0$, let,

$$\tilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log\,{\rm Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\,\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha - \frac{1}{\alpha-1}\log\,{\rm Tr}\,\rho \tag{1}$$

for $\alpha \in (0,1)\cup(1,\infty)$, with the convention that $\tilde{D}_\alpha(\rho\|\sigma)=\infty$ if $\alpha>1$ and $\ker\sigma \not\subset \ker\rho$. The quantity (1) is called the *quantum Rényi divergence* in [9] or the *sandwiched Rényi relative entropy* in [11], and is extended to $\alpha=1$ by continuity, to obtain the von Neumann relative entropy. The limiting cases $\alpha \downarrow 0$ and $\alpha \to \infty$ have also been studied in [4, 2] and [9], respectively. The sandwiched Rényi relative entropy has several desirable properties: amongst others, if $\alpha \geq \frac{1}{2}$, it is monotone under completely positive trace preserving maps [9, 11, 3, 6]. This property was successfully used in studying the strong converse properties of the channel capacity [11, 8] and the quantum hypothesis testing problem [7].

Now we confine our attention to the case when both ρ and σ are faithful density operators that belong to the *quantum state space* $\mathcal{S}(\mathcal{H}) := \{\rho \in \mathcal{L}_{++}(\mathcal{H}) \mid {\rm Tr}\,\rho = 1\}$. In this case there is no difficulty in extending the quantity (1) to the region $\alpha<0$. However, it does not seem to give a reasonable measure of information for $\alpha<0$ [10], since it takes negative values. Motivated by this fact, we study the "rescaled" sandwiched Rényi relative entropy:

$$D_\alpha(\rho\|\sigma) := \frac{1}{\alpha}\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha(\alpha-1)}\log\,{\rm Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\,\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha \tag{2}$$

for $\alpha \in \mathbb{R}\backslash\{0,1\}$ and $\rho,\sigma \in \mathcal{S}(\mathcal{H})$, and is extended to $\alpha=1$ by continuity. We shall call the quantity (2) as the *sandwiched Rényi α-divergence*. In particular, we are interested in the information geometrical structure [1, 5] induced from (2) on the quantum state space $\mathcal{S}(\mathcal{H})$.

Theorem 1. *The induced Riemannian metric $g^{(D_\alpha)}$ is monotone under completely positive trace preserving maps if and only if $\alpha \in (-\infty, -1] \cup [\frac{1}{2}, \infty)$.*

As a by-product, we arrive at the following corollary, the latter part of which was first observed by numerical evaluation [9].

Corollary 2. *The sandwiched Rényi α-divergence $D_\alpha(\rho||\sigma)$ is not monotone under completely positive trace preserving maps if $\alpha \in (-1, 0) \cup (0, \frac{1}{2})$. Consequently, the original sandwiched Rényi relative entropy $\tilde{D}_\alpha(\rho||\sigma)$ is not monotone if $\alpha \in (0, \frac{1}{2})$.*

We also studied the dualistic structure $(g^{(D_\alpha)}, \nabla^{(D_\alpha)}, \nabla^{(D_\alpha)*})$ on the quantum state space $\mathcal{S}(\mathcal{H})$, and obtained the following

Theorem 3. *The quantum statistical manifold $(\mathcal{S}(\mathcal{H}), g^{(D_\alpha)}, \nabla^{(D_\alpha)}, \nabla^{(D_\alpha)*})$ is dually flat if and only if $\alpha = 1$.*

Keywords: quantum Rényi α-divergence, monotone metric, dually flatness

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