Symmetry condition for partially factorizable discrete distributions

Aleš Kuběna
ÚTIA, Czech Academy of Sciences
Prague, Czech Republic
e-mail: kubena@utia.cas.cz

Definition

For a hypergraph $\mathcal{A} \subseteq 2^I$, $|I| = n$, we define a probability manifold of everywhere-nonzero $\mathcal{A}$-factorizable distributions on $\Omega_I = \{0,1\}^I$,

$$\mathcal{P}_\mathcal{A} = \{\mathbf{p} : \exists_{\theta_Z; Z\in\mathcal{A}} \forall_{X\subseteq I} \ \mathbf{p}(X = (1,\ldots,1)) = \prod_{Z\in\mathcal{A}\cap 2^X} \theta_Z\}$$

Let $\mathcal{P}_k = \mathcal{P}_{\{Z:|Z|=k\}}$ Any ordering $\varphi : I \to \{0,\ldots,n-1\}$ define a projection operator

$$m_\varphi : \mathcal{P}_\mathcal{A} \to \mathcal{P}_1, \quad m_\varphi(\mathbf{p})|_k = \frac{\mathbf{p}(\{i : \varphi(i) \leqq k\})}{\mathbf{p}(\{i : \varphi(i) < k\})}.$$

For a permutation group $G \le S_n$ let $m_{\varphi G}(\mathbf{p})|_k = GeoMean_{\pi\in G}(m_{\pi\cdot\varphi}(\mathbf{p})|_k)$.

Results

We proved that there exist an algebraic condition for the permutation group G, for which the operator $m_{\varphi G}$ is defined unambiguously on $\mathcal{P}_\mathcal{A}$, that is, $m_{\varphi G}$ is independent of the original order φ:

Let $sOrb(G)$ be the set of all orbits of all subgroups of G. We claim that $m_{\varphi G} : \mathcal{P}_\mathcal{A} \to \mathcal{P}_1$ is independent of the original order φ iff $\mathcal{A} \subseteq sOrb(G)$. Such permutation groups G can be classified for $\mathcal{A}_k = \begin{pmatrix} I \\ k \end{pmatrix}$, $k > 2$. The case $k = 2$ is hard.

In the special case $\mathcal{A} = 2^I$, the condition $\mathcal{A} \subseteq sOrb(G)$ is satisfied only for the full symmetric group S_n for all n, alternating group A_n for $n \ge 4$, and the image of a non-standard embedding $S_5 \to S_6$ for $n = 6$. If $\mathcal{A} = \mathcal{A}_k$, then a subgroup $G \le S_n$ different from S_n and A_n that satisfies $\mathcal{A} \subseteq sOrb(G)$ only exists for $k \le 6$ or $k \ge n-3$.

On the other hand, the trivial group $G = \{e\}$ guarantees φ-independency only for a system of n independent Bernoulli distributions.

For a general permutation group $G \leq S_n$, the set of all operators $\{m_{\varphi G} : \varphi \in I^n\}$ forms a convex set with dimension

$$\left[[u^k] \left(e^{\sum_{i=1}^n u^i \frac{\partial}{\partial x_i}} u \frac{\partial}{\partial x_1} - e^{\sum_{i=1}^n u^i \frac{\partial}{\partial x_i}} \right) \right] Z_G(1, 1, \ldots, 1)$$

where Z_G is the cycle index of the permutation group G [1] and $e^{\sum_{i=1}^n u^i \frac{\partial}{\partial x_i}}$ is applied as a formal operator.

References

[1] Cameron, P. (1999) Permutation Groups, Cambridge University Press, ISBN 0-521-65379

[2] Giblisco, P. & all (ed.), (2009) Algebraic and geometric method in statistic, Cambridge University Press, ISBN 9780511642401