Revisit to the autoparallelity and the canonical divergence for dually flat spaces

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Let (S, g, ∇, ∇^*) be a finite-dimensional dually flat space with the canonical divergence D, including the representative example where S is an exponential family, g is the Fisher metric, ∇ and ∇^* are the e, m-connections and D is the relative entropy (KL divergence). A submanifold M is said to be ∇-autoparallel (∇^*-autoparallel, resp.) when M forms an open subset of an affine subspace in the coordinate space of a ∇-affine (∇^*-affine, resp.) coordinate system. Note that, when M is either ∇-autoparallel or ∇^*-autoparallel, M itself becomes dually flat. For a submanifold M of S, the following three conditions are shown to be equivalent.

1. There exists a ∇-autoparallel submanifold K of S such that M is a ∇^*- autoparallel submanifold of K.
2. There exists a ∇^*-autoparallel submanifold K of S such that M is a ∇- autoparallel submanifold of K.
3. M is a dually flat space for which the canonical divergence is the restriction $D|_{M\times M}$ of D.

In addition, a submanifold M satisfying (1)-(3) can be represented as $M = K_1 \cap K_2$ by a ∇-autoparallel K_1 and a ∇^*-autoparallel K_2. An important example is given by $S = P(\mathcal{X}^n)$ (the set of positive n-joint distributions) with K_1 being the exponential family consisting of markovian distributions and K_2 being the mixture family consisting of stationary distributions, so that the set $M = K_1 \cap K_2$ of stationary markovian distributions becomes dually flat with the relative entropy as the canonical divergence.