QUANTUM MEASUREMENT AS A CONDITIONAL EXPECTATION

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The state of a quantum system is described by a wave function, this is a normalized element ψ of a Hilbert space, or by a density matrix ρ, which is a trace-class operator with trace 1 and non-negative eigenvalues. Typical for a quantum measurement is that it does *not* reveal the state of the system up to some measurement error. This is a major deviation from the situation in classical systems. If one measures the speed of say a car then the result is a number which usually is a good approximation of the actual speed of the car. This is not the case for quantum measurements. The measurement process itself must be described in quantum-mechanical terms. The coupling between the system at hand and the measurement apparatus fixes an orthonormal basis $(e_n)_n$ in the Hilbert space of quantum states. The outcome of the experiment is then that the state of the system is e_n with probability $|\langle e_n|\psi\rangle|^2$, where $\langle\cdot|\cdot\rangle$ denotes the inner product of the Hilbert space. This phenomenon is known as the *collapse of the wave function*. From these probabilities one can then try to reconstruct the original wave function ψ by repeating the measurement under identical initial conditions.

The point of view here is that the experimental setup necessarily introduces a condition on the outcomes of the experiment. Conditional expectations have been studied in quantum probability theory [4]. The origin of these studies is the discovery [1] of a link with the Tomita-Takesaki theory, which describes one-parameter groups of automorphisms of von Neumann algebras. The condition that the outcome of the experiment is an element of an orthonormal basis is a conditional expectation in this mathematical sense.

The probabilities $|\langle e_n|\psi\rangle|^2$ can now be explained in geometrical terms. The quantum analogue of the Kullback-Leibler divergence is given by

$$D(\sigma||\rho) = \mathrm{Tr}\sigma\ln\sigma - \mathrm{Tr}\sigma\ln\rho. \qquad (1)$$

Here, σ and ρ are density matrices. Assume now that σ is the orthogonal projection onto the multiples of the wave function ψ and that ρ is conditioned to be diagonal in the given basis $(e_n)_n$. Then the divergence $D(\sigma||\rho)$ is minimal when the diagonal elements of the matrix ρ equal the probabilities $|\langle e_n|\psi\rangle|^2$. One concludes that the experimental outcome is given by the density matrix which minimizes the divergence $D(\sigma||\rho)$ under the condition that the matrix ρ is diagonal.This minimization process is equivalent to an orthogonal projection onto the manifold of diagonal density matrices.

At the end of the previous century physicists succeeded in devising experiments which avoid the collapse of the wave function. In some cases, thousands of consecutive measurements are possible [3] before the collapse of the wave function is reached. Such measurements are now referred to as *weak measurements*. In a typical setup the system under study is weakly coupled to a second quantum system. On the latter strong measurements are performed as usual. By keeping the coupling between the two subsystems very weak the system of interest is not too much disturbed by the measurements. In addition, by a proper choice of basis vectors the sensitivity of the experiment can be increased [2]. The latter can be understood from the fact that the divergence function diverges at the borders of the manifold [5].

Keywords: Quantum probability theory, quantum conditional expectations, quantum divergence, weak measurements.

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