On Robertson-type uncertainty principles

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Basic Notations in Quantum Mechanics

State space: the set of $n \times n$ positive definite trace one matrices ($\mathcal{M}_n^1$).
Observables: $n \times n$ self adjoint matrices ($M_{n,\mathrm{sa}}$).

For given state $D \in \mathcal{M}_n^1$ and observables $A, B \in M_{n,\mathrm{sa}}$
expectation value: $\mathrm{Tr}\,(DA)$;
normalization of A: $A_0 = A - \mathrm{Tr}(DA)I$; $(\mathrm{Tr}(DA_0) = 0)$
variance: $\mathrm{Var}_D(A) = \mathrm{Tr}\,(DA^2) - (\mathrm{Tr}\,(DA))^2$;
covariance: $\mathrm{Cov}_D(A,B) = \frac{1}{2}\Big(\mathrm{Tr}(DAB) + \mathrm{Tr}(DBA)\Big) - \mathrm{Tr}(DA)\,\mathrm{Tr}(DB)$.

Uncertainty Relations in Early Years

1927, Heisenberg: Defined the uncertainty of a Gaussian distribution f as its width D_f. The width of the Fourier transformation of f is denoted by $D_{\mathcal{F}(f)}$. The first formalization of the uncertainty principle was $D_f D_{\mathcal{F}(f)} =$ constant.

1927, Kennard: Observables A, B with $[A,B] = -\mathrm{i}$: $\mathrm{Var}_D(A)\,\mathrm{Var}_D(B) \geq \frac{1}{4}$.

1929, Robertson: For every observables A, B and state D: $\mathrm{Var}_D(A)\,\mathrm{Var}_D(B) \geq \frac{1}{4}|\,\mathrm{Tr}(D\,[A,B])|^2$.

1930, Scrödinger: $\mathrm{Var}_D(A)\,\mathrm{Var}_D(B) - \mathrm{Cov}_D(A,B)^2 \geq \frac{1}{4}|\mathrm{Tr}(D\,[A,B])|^2$.

1934, Robertson: For every set of observables $(A_i)_{1,\dots,N}$

$$\det\left(\left[\mathrm{Cov}_D(A_h, A_j)\right]_{h,j=1,\dots,N}\right) \geq \det\left(\left[-\frac{\mathrm{i}}{2}\mathrm{Tr}(D\,[A_h, A_j])\right]_{h,j=1,\dots,N}\right).$$

(For $N = 2$ gives the Scrödinger uncertainty relation.)

Quantum Fisher Information

$\mathcal{F}_{\mathrm{op}}$: set of operator monotone functions $f : \mathbb{R}^+ \to \mathbb{R}$ with properties $f(x) = xf(x^{-1})$ and $f(1) = 1$.

Examples in $\mathcal{F}_{\mathrm{op}}$: $f_{\mathrm{RLD}}(x) = \frac{2x}{1+x}$, $f_{\mathrm{SLD}}(x) = \frac{1+x}{2}$, $f_{\mathrm{WY}}(x) = \left(\frac{1+\sqrt{x}}{2}\right)^2$, $f_{\mathrm{KM}}(x) = \frac{x-1}{\log x}$.

Regular and non-regular elements: $\mathcal{F}_{\mathrm{op}}^{\mathrm{r}} = \left\{f \in \mathcal{F}_{\mathrm{op}} | f(0) \neq 0\right\}$ and $\mathcal{F}_{\mathrm{op}}^{\mathrm{n}} = \left\{f \in \mathcal{F}_{\mathrm{op}} | f(0) = 0\right\}$.

Theorem [*Gibilisco, Hansen, Isola*]. *The map*

$$\mathcal{F}_{\mathrm{op}}^{\mathrm{r}} \to \mathcal{F}_{\mathrm{op}}^{\mathrm{n}} \qquad f \mapsto \tilde{f}(x) = \frac{1}{2}\left[(1+x) - (1-x)^2\frac{f(0)}{f(x)}\right]$$

is a bijection.

For every $f \in \mathcal{F}_{\mathrm{op}}$ introduce the notation $m_f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $m_f = yf\left(\frac{x}{y}\right)$.
(The reciprocal of m_f is the Chentsov–Morozova function.)

Theorem [*Petz*]. *In quantum setting there is a bijective correspondence between Fisher informations and functions in $f \in \mathcal{F}_{\mathrm{op}}$. For every $f \in \mathcal{F}_{\mathrm{op}}$ the Fisher information is given by*

$$\langle A, B\rangle_{D,f} = \mathrm{Tr}\left(Am_f\left(L_D, R_D\right)^{-1}(B)\right), \tag{1}$$

where $L_D(X) = DX$, $R_D(X) = XD$.
$\Longrightarrow$ For every $f \in \mathcal{F}_{\mathrm{op}}$ $(\mathcal{M}_n, \langle\cdot,\cdot\rangle_{\cdot,f})$ is a Riemannian manifold.

Covariances

For two observables $A, B \in M_{n,\mathrm{sa}}$, state $D \in \mathcal{M}_n^1$ and function $f \in \mathcal{F}_{\mathrm{op}}$ we define
covariance:

$$\mathrm{Cov}_D(A,B) = \frac{1}{2}\left(\mathrm{Tr}(DAB) + \mathrm{Tr}(DBA)\right) - \mathrm{Tr}(DA)\,\mathrm{Tr}(DB);$$

quantum f-covariance: [introduced by Petz]

$$\mathrm{Cov}_D^f(A,B) = \mathrm{Tr}\left(Af(L_{n,D}R_{n,D}^{-1})R_{n,D}(B)\right);$$

antisymmetric f-covariance:

$$\mathrm{qCov}_{D,f}^{as}(A,B) = \frac{f(0)}{2}\left\langle \mathrm{i}\,[D,A]\,, \mathrm{i}\,[D,B]\right\rangle_{D,f};$$

symmetric f-covariance:

$$\mathrm{qCov}_{D,f}^{s}(A,B) = \frac{f(0)}{2}\left\langle \{D,A\}\,, \{D,B\}\right\rangle_{D,f},$$

where $[.,.]$ is the commutator of matrices and $\{.,.\}$ denotes the anticommutator respectively.

For a fixed density matrix $D \in \mathcal{M}_n^1$, function $f \in \mathcal{F}_{\mathrm{op}}$ and an N-tuple of nonzero matrices $(A^{(k)})_{k=1,\dots,N} \in M_{n,\mathrm{sa}}$ we define the following $N \times N$ matrices Cov_D, Cov_D^f, $\mathrm{qCov}_{D,f}^{as}$ and $\mathrm{qCov}_{D,f}^{s}$ with entries

$$[\mathrm{Cov}_D]_{ij} = \mathrm{Cov}_D(A_0^{(i)}, A_0^{(j)}) \qquad \left[\mathrm{Cov}_D^f\right]_{ij} = \mathrm{Cov}_D^f(A_0^{(i)}, A_0^{(j)})$$
$$\left[\mathrm{qCov}_{D,f}^{as}\right]_{ij} = \mathrm{qCov}_{D,f}^{as}(A_0^{(i)}, A_0^{(j)}) \qquad \left[\mathrm{qCov}_{D,f}^{s}\right]_{ij} = \mathrm{qCov}_{D,f}^{s}(A_0^{(i)}, A_0^{(j)}).$$

Mathematical Toolbox

Petz's scalar produc (1) can be extended: Define

$$\mathcal{C}_{\mathcal{M}} = \left\{ g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \;\middle|\; \begin{array}{l} g \text{ is a symmetric smooth function, with analytical} \\ \text{extension defined on a neighborhood of } \mathbb{R}^+ \times \mathbb{R}^+ \end{array} \right\}.$$

Fix a function $g \in \mathcal{C}_{\mathcal{M}}$. Define for every $D \in \mathcal{M}_n$ and for every $A, B \in M_{n,\mathrm{sa}}$

$$(A,B)_{D,g} = \mathrm{Tr}\left(Ag(L_{n,D}, R_{n,D})(B)\right).$$

$\Longrightarrow$ For every $g \in \mathcal{C}_{\mathcal{M}}$ $(\mathcal{M}_n, (\cdot,\cdot)_{\cdot,g})$ is a Riemannian manifold.

Connection to Petz's scalar product: For $f \in \mathcal{F}_{\mathrm{op}}$ define $g(x,y) = \frac{1}{yf\left(\frac{x}{y}\right)}$, then we have

$$\langle A, B\rangle_{D,f} = (A,B)_{D,g} \qquad \forall D \in \mathcal{M}_n,\ \forall A, B \in M_{n,\mathrm{sa}}.$$

Theorem [*Andai, Lovas*]. *Consider a density matrix $D \in \mathcal{M}_n^1$, an N-tuple of observables $(A^{(k)})_{k=1,\dots,N}$ and functions $g_1, g_2 \in \mathcal{C}_{\mathcal{M}}$ such that*

$$g_1(x,y) \geq g_2(x,y) \qquad \forall x, y \in \mathbb{R}^+.$$

Define the $N \times N$ matrices $\mathfrak{Cov}_{D,g_1}$ and $\mathfrak{Cov}_{D,g_2}$ with entries $\left[\mathfrak{Cov}_{D,g_k}\right]_{ij} = (A_0^{(i)}, A_0^{(j)})_{D,g_k}$ $(k = 1, 2)$. Then

$$\det(\mathfrak{Cov}_{D,g_1}) \geq \det(\mathfrak{Cov}_{D,g_2}) + \det(\mathfrak{Cov}_{D,g_1} - \mathfrak{Cov}_{D,g_2})$$

holds.

Uncertainty Relations Nowadays

Gibilisco and Isola in 2006 conjectured the inequality $\det(\mathrm{Cov}_D) \geq \det(\mathrm{qCov}_{D,f}^{as})$.
Which was based on numerous partial results for very specific f functions for few (generally 2) observables and the inequalities were expressed in different form. The conjecture was proved by Andai and Gibilisco, Imparato and Isola in 2008.

We have found a more accurate inequality:
Theorem [*Lovas, Andai*]. *For any operator monotone function $f \in \mathcal{F}_{\mathrm{op}}$ at every state $D \in \mathcal{M}_n^1$ for every N-tuple of observables $(A^{(k)})_{k=1,\dots,N}$ we have for the covariance matrices*

$$\det(\mathrm{Cov}_D) \geq \det(\mathrm{qCov}_{D,f}^{s}) \geq \det(\mathrm{qCov}_{D,f}^{as}).$$

We have an estimation for the gap between the symmetric and antisymmetric covariance:

Theorem [*Lovas, Andai*]. *Using the same notation as in the previous Theorem we have*

$$\det(\mathrm{qCov}_{D,f}^{s}) - \det(\mathrm{qCov}_{D,f}^{as}) \geq (2f(0))^N \det(\mathrm{Cov}_D^{f_{RLD}}),$$

where $f_{RLD}(x) = \frac{2x}{1+x}$.

Moreover we have shown that the symmetric covariance generated by the function $f_{opt}(x) = \frac{1}{2}\left(\frac{1+x}{2} + \frac{2x}{1+x}\right)$ is universal in the following sense:

Theorem [*Lovas, Andai*]. *For every function $g \in \mathcal{F}_{\mathrm{op}}$ the inequality*

$$\det(\mathrm{qCov}_{D,f_{opt}}^{s}) \geq \det(\mathrm{qCov}_{D,g}^{as})$$

holds and f_{opt} gives the best upper bound in $\mathcal{F}_{\mathrm{op}}$.