QUANTUM MEASUREMENT AS A CONDITIONAL EXPECTATION

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Quantum measurements

The state of a quantum system is described by a wave function ψ
This is a normalized element of a Hilbert space $\mathcal{H}$

A measurement does not reveal ψ completely
Rather it fixes an orthonormal basis $(e_n)_n$ and
returns n with probability $|\langle e_n|\psi\rangle|^2$

The result can be encoded as a *density matrix* $\rho = \sum_n \lambda_n |e_n\rangle\,\langle e_n|$
with $\lambda_n = |\langle e_n|\psi\rangle|^2$ and
$|e_n\rangle\langle e_n|$ is the orthogonal projection onto $\mathbb{C}e_n$.

Collapse of the wave function

The outcome of the measurement is one of the basis vectors e_n
The system is forced into the state e_n by the measurement
This phenomenon is known as the *collapse of the wave function*

Viewpoint

The measurement imposes a condition on the system

The experimental measurement is modeled by means of a conditional expectation
Let $\sigma = |\psi\rangle\langle\psi|$ denote the orthogonal projection onto $\mathbb{C}\psi$
Note that ρ is the diagonal part of the matrix σ in the basis $(e_n)_n$
The map $\sigma \to \rho = \mathrm{diag}(\sigma)$ is a conditional expectation in the sense of Petz

Quantum mechanics is a deterministic theory

However, results of experiments necessarily are indeterministic because of information being incomplete

Quantum conditional expectations

Accardi (1981), Petz (2008)

A conditional expectation is a linear map $E : \mathcal{B} \to \mathcal{A}$ from an algebra $\mathcal{B}$ of bounded operators to a subalgebra $\mathcal{A}$ satisfying

- $\mathbb{I}$ belongs to $\mathcal{A}$ and $E(\mathbb{I}) = \mathbb{I}$.
- If $A \in \mathcal{A}$ then also $A^\dagger \in \mathcal{A}$.
- If B is positive then also $E(B)$ is positive.
- $E(AB) = AE(B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

The conditional expectation satisfies a Pythagorean relation
The proof is based on Tomita-Takesaki theory

Reference

J. Naudts and B. Anthonis, *Extension of Information Geometry to Non-statistical Systems: Some Examples,* in: Geometric Science of Information, GSI 2015 LNCS proceedings, F. Nielsen and F. Barbaresco eds., (Springer, 2015)

Geometric description

The Kulback-Leibler divergence $D(\sigma||\rho)$ (relative entropy) of two density matrices is given by

$$D(\sigma||\rho) = \mathrm{Tr}\sigma\ln\sigma - \mathrm{Tr}\sigma\ln\rho$$

Minimize $D(\sigma||\rho)$ under the condition that ρ is diagonal in the given basis $(e_n)_n$.

This results in $\rho = \mathrm{diag}(\sigma)$ with $\sigma = |\psi\rangle\langle\psi|$

Calculation

$$\mathcal{L} = D(\sigma||\rho) + \alpha\sum_n \lambda_n = -\sum_n |\langle e_n|\psi\rangle|^2 \ln\lambda_n + \alpha\sum_n \lambda_n$$

$$0 = \frac{\partial}{\partial\lambda_n}\mathcal{L} = -\frac{1}{\lambda_n}|\langle e_n|\psi\rangle|^2 + \alpha$$

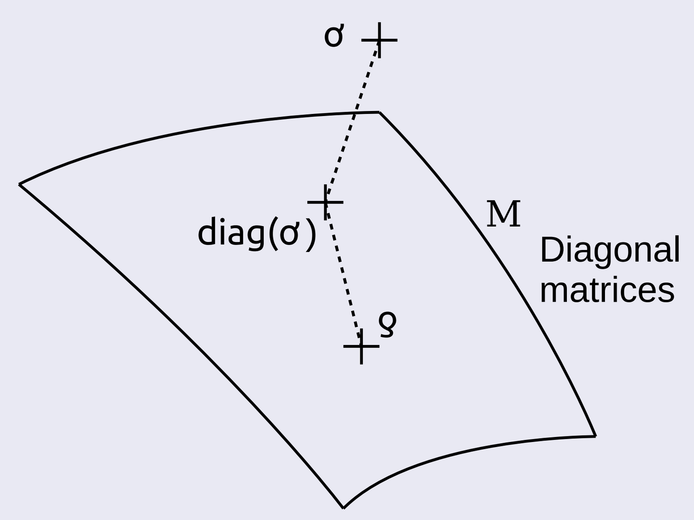
Normalization implies $\alpha = 1$ and $\lambda_n = |\langle e_n|\psi\rangle|^2$

A Pythagorean relation

$$D(\sigma||\rho) = D(\sigma||\mathrm{diag}(\sigma)) + D(\mathrm{diag}(\sigma)||\rho)$$

holds for all diagonal ρ

Hence, $\mathrm{diag}(\sigma)$ is the *orthogonal* projection of σ onto the manifold of diagonal matrices



Weak measurements

First discussed by Aharonov, Albert, Vaidman (1988)

In recent experiments the collapse of the wave function is avoided by weakly coupling the system of interest to a second quantum system; traditional measurements involving collapse of the wave function are performed on the latter

For instance Guerlin et al (2007), Nobel price 2012 S. Haroche (together with D. Wineland)

The sensitivity of weak measurements can be understood in geometric terms

The coupling with the second system determines the basis $(e_n)_n$
Choose this basis so that $\mathrm{diag}(\sigma)$ is in the border of the domain of definition of $D(\sigma||\rho)$
Then $D(\sigma||\mathrm{diag}(\sigma))$ is close to divergence and very sensitive to small changes of σ