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CONSTANT CURVATURE CONNECTIONS ON STATISTICAL MODELS

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ФИНАНСОВЫЙ УНИВЕРСИТЕТ ПРИ ПРАВИТЕЛЬСТВЕ РОССИЙСКОЙ ФЕДЕРАЦИИ

ABSTRACT

We discuss statistical manifolds [1] with connections of constant α- curvature. If the statistical manifold has some α- connection of constant curvature then it is a conjugate symmetric manifold [2]. But the converse is not true [3]. Statistical models on families of probability distributions provide important examples of above-mentioned connections. Normal statistical models have the constant α- curvature $k^{(\alpha)}=\frac{\alpha^2-1}{2}$ for any parameter α. **We obtain** that the **Pareto** two-dimensional statistical models [4] has such a structure: **each of its α- connection has the constant curvature $(-2\alpha-2)$**. The logistic two-dimensional statistical model [5] has the constant 2-curvature $k^{(2)}=-\frac{162}{(\pi^2+3)^2}$, so it is a conjugate symmetric manifold [6]. **We consider** a **Weibul**l two-dimensional statistical model: **its 1-connection has the constant curvature**

$$k^{(1)}=\frac{12\pi^2\gamma-144\gamma+72}{\pi^4},$$

where $\gamma=\lim_{n\to\infty}(\sum_{k=1}^{\infty}\frac{1}{k}-\log n)$ is Euler-Mascheroni constant. Thus, the **Weibull model is a conjugate symmetric**.
We compare the values of **α - curvatures** for the different statistical models.

STATISTICAL MANIFOLD

Let M be a smooth manifold, $\dim M = n$,
$\langle\cdot,\cdot\rangle = g$ be a Riemannian metrics,
K be a (2,1) - tensor such that **(1)** $K(X,Y)=K(Y,X)$; **(2)** $\langle K(X,Y),Z\rangle=\langle Y,K(X,Z)\rangle$, where
X,Y,Z are vector fields on M.
Then a triple (M,g,K) is **a statistical manifold**.
If D is the metric connection, i.e.$Dg=0$, α is a real-valued parameter,
then the linear connections of 1 - parameter family $\nabla^\alpha = D+\alpha\cdot K$ are called **α - connections**.

STATISTICAL MODEL

Let $S=\{P_{\theta^i}\mid i=1,\dots,n\}$ be a family of probability distributions on a measurable space with a probability measure P, $\partial_i=\frac{\partial}{\partial\theta^i}$
$\log p(x\mid\theta^i)$ denotes the natural logarithm of the likelihood function.
Then S is called a n - dimensional **statistical model**.
The Fisher information matrix $I_{ij}(\theta)=\int\partial_i\log p\cdot\partial_j\log p\cdot pdP$
and the components $K_{ijk}(\theta)=-\frac{1}{2}\int\partial_i\log p\cdot\partial_j\log p\cdot\partial_k\log p\cdot pdP$ $(j,k=1,\dots,n)$
give the structure of the statistical manifold (P,I_{ij},K_{ij}) on S.
The covariant components of α - connections named **Amari-Chentsov connections are**
$\Gamma^{(\alpha)}_{ijk}(\theta)=\int(\partial_i\partial_j\log p\cdot\partial_k\log p+\frac{1-\alpha}{2}\partial_i\log p\cdot\partial_j\log p\cdot\partial_k\log p)\cdot pdP$

NORMAL 2-DIM MODEL

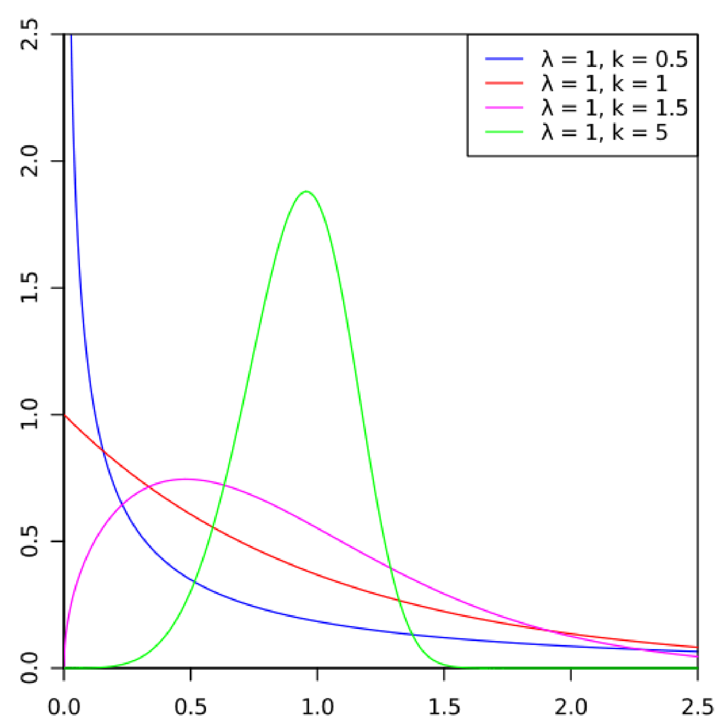
$p(x\mid\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{(x-\mu)^2}{2\sigma^2}\};\mu=\theta^1,\sigma=\theta^2>0;$

$$I_{ij}=\begin{pmatrix}\frac{1}{\sigma^2}&0\\0&\frac{1}{\sigma^2}\end{pmatrix}$$

$$\boxed{k^{(\alpha)}=\frac{\alpha^2-1}{2}}$$

WEIBULL 2-DIM MODEL

$$p(x\mid\lambda,k)=(\frac{k}{\lambda})\cdot(\frac{x}{\lambda})^{k-1}\cdot\exp\{-(\frac{x}{\lambda})^k\};\lambda=\theta^1>0,k=\theta^2>0;x>0.$$



The distribution function is $F(x\mid\lambda,k)=1-\exp\{-(\frac{x}{\lambda})^k\}$, logarithm of the likelihood function and its partial derivatives are
$\ln p(x\mid\lambda,k)=\ln(\frac{k}{\lambda})+(k-1)\cdot\ln(\frac{x}{\lambda})-(\frac{x}{\lambda})^k$; $\partial_1\ln p(x\mid\lambda,k)=-\frac{k}{\lambda}+\frac{k}{\lambda^{k+1}}x^k$;
$\partial_2\ln p(x\mid\lambda,k)=\frac{1}{k}+(1-(\frac{x}{\lambda})^k)\cdot\ln(\frac{x}{\lambda})$. We obtain the non-zero components of the structure as

$$I=\begin{pmatrix}(\frac{k}{\lambda})^2&\frac{\gamma-1}{\lambda}\\\frac{\gamma-1}{\lambda}&\frac{\pi^2+6\gamma^2-12\gamma+6}{6k^2}\end{pmatrix}$$

$\det I_{ij}=\frac{\pi^2}{6\lambda^2}$;
$K_{111}=-(\frac{k}{\lambda})^3$, $K_{112}=K_{121}=K_{211}=\frac{(2-\gamma)k}{\lambda^2}$, $K_{122}=K_{212}=K_{221}=-\frac{\pi^2+6\gamma^2-24\gamma+12}{6\lambda k}$,
$K_{222}=\frac{\pi^2(2-\gamma)-4\zeta(3)-2\gamma^3+12\gamma^2-12\gamma+2}{2k^3}$.
Consider the case $\alpha=1$. Then we have the non-zero components ${}^{(1)}\Gamma^i_{jk}$ of **Amari-Chentsov connections** on the **Weibull model** as
${}^{(1)}\Gamma^1_{11}=-\frac{k+1}{\lambda}$, ${}^{(1)}\Gamma^1_{12}={}^{(1)}\Gamma^1_{21}=\frac{2\pi^2-6\gamma+6}{\pi^2k}$, ${}^{(1)}\Gamma^2_{12}={}^{(1)}\Gamma^2_{21}=-\frac{(\pi^2-6)k}{\pi^2\lambda}$,
${}^{(1)}\Gamma^1_{22}=-\frac{\pi^4-12\pi+6\pi^2\gamma(2-\gamma)+72(\gamma-1)^2-72\zeta(3)(\gamma-1)}{6\pi^2k^3}\lambda$, ${}^{(1)}\Gamma^2_{22}=-\frac{2\pi^2(2-\gamma)-12\zeta(3)+12\gamma-12}{\pi^2k}$.
Theorem 4.Weibull two-dimensional statistical model is a conjugate symmetric statistical manifold. Its **1-connection has the constant curvature**

$$k^{(1)}=\frac{12\pi^2\gamma-144\gamma+72}{\pi^4},$$

where $\gamma=\lim_{n\to\infty}(\sum_{k=1}^{\infty}\frac{1}{k}-\log n)$ is Euler-Mascheroni constant.

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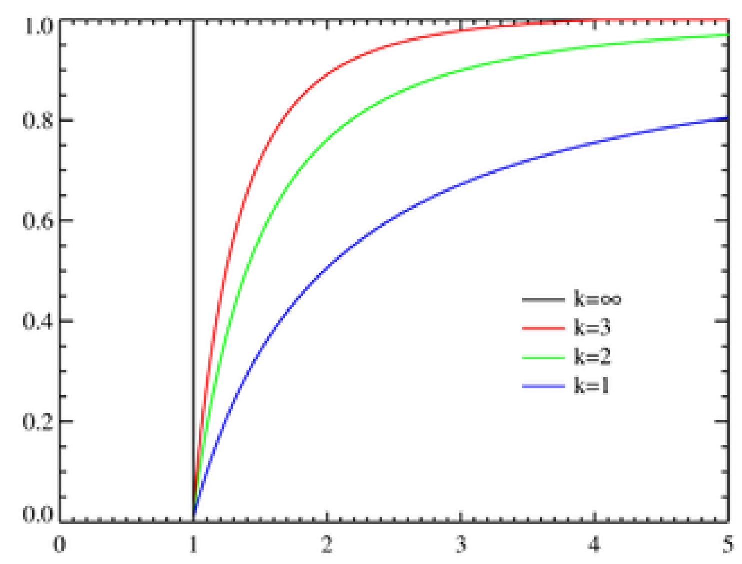
CONJUGATE STATISTICAL MANIFOLD

Denote R^α_{XY} a curvature operator of ∇^α, Ric$^\alpha$ a Ricci tensor of ∇^α
ω_g a volume element associated to the metrics.
We call (M,g,K) a **conjugate symmetric manifold**, when $R^\alpha_{XY}g=0$ for any parameter α.
This is equivalent to the equality R^α=$R^{-\alpha}$ for any dual α - connections.
Theorem 1. If **some** α - connection has a constant α - curvature $k^{(\alpha)}$, i.e.
$R^\alpha(X,Y)Z$=$k^\alpha(\langle Y,Z\rangle X-\langle X,Z\rangle Y)$, then (M,g,K) is **a conjugate symmetric** manifold.
Theorem 2. If (M,g,K) is **a conjugate symmetric** manifold with α - connections which are
(1) equiprojective, i.e. $R^\alpha(X,Y)Z=\frac{1}{n-1}(\text{Ric}^\alpha(X,Z)Y-\text{Ric}^\alpha(Y,Z)X)$,
(2) strongly compatible with the metrics g, i.e. $(\nabla^\alpha_Xg)(Y,Z)$=$(\nabla^\alpha_Yg)(X,Z)$; $\nabla^\alpha\omega_g=0$,
then (M,g,K) is a statistical manifold of a **constant α - curvature**.

PARETO 2-DIM MODEL

$p(x\mid a,\rho)=\rho a^\rho x^{-\rho-1}$; $a=\theta^1>0,\rho=\theta^2>0;x\geq a$.



The distribution function is $F(x\mid a,\rho)=1-(\frac{a}{x})^\rho$, the non-zero components of the structure are

$$I_{ij}=\begin{pmatrix}(\frac{\rho}{a})^2&0\\0&\frac{1}{\rho^2}\end{pmatrix}$$

$K_{111}=-(\frac{1}{2})\cdot(\frac{\rho}{a})^3$, $K_{112}=K_{212}=K_{221}=-\frac{1}{2a\rho}$, $K_{222}=\frac{1}{\rho^3}$.
Then we have the non-zero components ${}^{(\alpha)}\Gamma^i_{jk}$ of **Amari-Chentsov connections** on the **Pareto model** as,
${}^{(\alpha)}\Gamma^1_{11}=-\frac{2+\alpha\rho}{2a}$, ${}^{(\alpha)}\Gamma^2_{11}=-\frac{\rho^3}{a^2}$, ${}^{(\alpha)}\Gamma^1_{12}={}^{(\alpha)}\Gamma^1_{21}=\frac{1}{\rho}$, ${}^{(\alpha)}\Gamma^2_{12}={}^{(\alpha)}\Gamma^2_{21}=-\frac{\alpha\rho}{2a}$, ${}^{(\alpha)}\Gamma^1_{22}=-\frac{\alpha a}{2\rho^3}$,
${}^{(\alpha)}\Gamma^2_{22}=-\frac{1-\alpha}{\rho}$.
Theorem 3. Pareto two-dimensional statistical model is a statistical manifold of a **constant α-curvature**

$$\boxed{k^{(\alpha)}=-2\alpha-2}$$

In particular, $k^{(0)}=-2$, i.e. Riemannian metrics on such manifold has the constant negative curvature.

LOGISTIC 2-DIM MODEL

The distribution function is $F(x\mid a,b)=\frac{1}{(1+\exp(-ax-b))}$, the non-zero components of the structure are

$$I_{ij}=\frac{1}{3a^2}\begin{pmatrix}b^2+\frac{\pi^2}{3}+1&-ab\\-ab&a^2\end{pmatrix}$$

$K_{111}=\frac{1}{6a^3}$, $K_{122}=K_{212}=K_{211}=\frac{1}{6a}$, $K_{211}=K_{112}=K_{121}=-\frac{b}{3a^2}$
Consider the case $\alpha=2$. Then we have the non-zero components ${}^{(2)}\Gamma^i_{jk}$ of Amari-Chentsov connections on the logistic model as
${}^{(2)}\Gamma^1_{11}=\frac{2\pi^2-3}{(\pi^2+3)a}$, ${}^{(2)}\Gamma^2_{11}=-\frac{9b}{(\pi^2+3)a^2}$, ${}^{(2)}\Gamma^2_{12}={}^{(2)}\Gamma^2_{21}=\frac{1}{a}$.
Hence the logistic model has the constant 2-curvature

$$\boxed{k^{(2)}=-\frac{162}{(\pi^2+3)^2}}$$

so it is a conjugate symmetric manifold [6].

USEFUL IMPROPER INTEGRALS

$\int_a^{+\infty}x^{-\rho-1}dx=\frac{1}{\rho a^\rho}$; $\int_a^{+\infty}\log x\cdot x^{-\rho-1}dx=\frac{1+\rho\log a}{\rho^2a^\rho}$; $\int_a^{+\infty}\log^2x\cdot x^{-\rho-1}dx=\frac{(1+\rho\log a)^2+1}{\rho^3a^\rho}$;
$\int_a^{+\infty}\log^3x\cdot x^{-\rho-1}dx=\frac{(1+\rho\log a)^3+3(1+\rho\log a)+2}{\rho^4a^\rho}$;
$\int_0^{+\infty}x^{nk-1}\cdot e^{-x^k}dx=\frac{(n-1)!}{k}$, $(n\in N,k>0)$; $\int_0^{+\infty}\log x\cdot x^{k-1}\cdot e^{-x^k}dx=-\frac{\gamma}{k^2}$;
$\int_0^{+\infty}\log^2x\cdot x^{k-1}\cdot e^{-x^k}dx=\frac{\pi^2+6\gamma^2}{6k^3}$;
$\int_0^{+\infty}\log^3x\cdot x^{k-1}\cdot e^{-x^k}dx=-\frac{4\zeta(3)+\gamma\pi^2+2\gamma^3}{2k^4}$,
where $n\in N,k>0$; $\gamma=\lim_{n\to\infty}(\sum_{k=1}^{\infty}\frac{1}{k}-\log n)\approx0,577\dots$ is Euler-Mascheroni constant,
$\zeta(3)=\sum_{k=1}^{\infty}\frac{1}{k^3}\approx1,202\dots$ is Apery's constant.

COMPARISON OF CURVATURES

∇^α	Normal	Logistic	Pareto	Weibull
$\alpha=0$	$-0,5$	$\approx-1,4$	-2	$\approx-1,22$
$\alpha=1$	0		-4	$\approx0,59$
$\alpha=2$	$1,5$	$\approx-0,98$	-6	