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Quantum entropy derived from first principles

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The non-commutative perspective

Consider a convex function f defined on a convex set $K \subseteq \mathbf{R}^n$. The perspective

$$\mathcal{P}_f(x,t) = tf(t^{-1}x)$$

is defined on the subset

$$L = \{(x,t) \in \mathbf{R}^n \times \mathbf{R} \mid t > 0 \quad \text{and} \quad t^{-1}x \in K\},$$

and it is a convex function of two variables.

We may extend the perspective of a function $f\colon (0,\infty) \to \mathbf{R}$ to positive definite matrices A and B by setting

$$\mathcal{P}_f(A,B) = B^{1/2} f(B^{-1/2}AB^{-1/2})B^{1/2}.$$

This definition makes sense also for non-commuting matrices (operators).

Convexity of the non-commutative perspective

Theorem

If $f : (0, \infty) \to \mathbf{R}$ is operator convex then the non-commutative perspective $\mathcal{P}_f$ is a convex function of two variables. Indeed, if

$$L = \lambda L_1 + (1-\lambda) L_2 \quad \text{and} \quad R = \lambda R_1 + (1-\lambda) R_2$$

for positive definite operators L_1, L_2 and R_1, R_2 then

$$\mathcal{P}_f(L, R) \leq \lambda \mathcal{P}_f(L_1, R_1) + (1-\lambda) \mathcal{P}_f(L_2, R_2)$$

for $\lambda \in [0, 1]$.

The perspective function,

$$\mathcal{P}_f(s, t) = t f(t^{-1} s) \qquad t, s > 0,$$

is in particular operator convex as a function of two variables.

A convex trace function

Theorem

Consider $n \times n$ matrices A and $n \times m$ matrices K. The trace function

$$\varphi(A) = -\operatorname{Tr} K^* AK \log(K^* AK) + \operatorname{Tr} K^*(A \log A)K$$

is convex in positive definite A for arbitrary K.

The function $f(t) = t \log t$ defined for $t > 0$ is operator convex. The perspective function,

$$\mathcal{P}_f(t, s) = sf(ts^{-1}) = t \log t - t \log s \qquad t, s > 0,$$

is therefore operator convex as a function of two variables, and this is equivalent to convexity of the map

$$\begin{aligned}(A, B) &\to \operatorname{Tr} K^* \big(L_{A \log A} - L_A R_{\log B}\big)(K)\\ &= \operatorname{Tr}\big(K^*(A \log A)K - K^* AK \log B\big) \qquad A, B > 0\end{aligned}$$

for every $K \in M_{n\times m}$.

The residual entropy

$$\varphi(A_1,\dots,A_k) = -\operatorname{Tr} A\log A + \sum_{i=1}^{k} \operatorname{Tr} A_i \log A_i \qquad A = A_1 + \cdots + A_k$$

is a convex function in positive definite $n \times n$ matrices $A_1,\dots,A_k$.

Proof: We apply the preceding theorem to block matrices of the form

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & A_k \end{pmatrix} \qquad \text{and} \qquad K = \begin{pmatrix} I & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ I & 0 & \cdots & 0 \end{pmatrix}.$$

The statement follows since

$$\varphi(A_1,\dots,A_k) = -\operatorname{Tr} K^*AK\log(K^*AK) + \operatorname{Tr} K^*(A\log A)K.$$

The von Neumann entropy gain over quantum channels

The entropy gain

$$\varphi(A) = S\big(\Phi(A)\big) - S(A)$$

over a quantum channel Φ is a convex function in A.

Proof: A quantum channel $\Phi\colon M_n \to M_m$ is of the form

$$\Phi(A) = \sum_{i=1}^{k} a_i^* A a_i$$

where the matrices $a_1, \ldots, a_k \in M_{n\times m}$ satisfy $a_1 a_1^* + \cdots + a_k a_k^* = 1$. We now apply the preceding theorem to the block matrices

$$\mathcal{A} = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & A \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_k & 0 & \cdots & 0 \end{pmatrix}.$$

Proof continued

The entry in the first row and the first column of the block matrix

$$-K^*AK\log(K^*AK) + K^*(A\log A)K$$

is calculated to $-\Phi(A)\log\Phi(A) + \Phi(A\log A)$. The entropic map

$$A \to S\big(\Phi(A)\big) + \operatorname{Tr}\Phi(A\log A) = S\big(\Phi(A)\big) - S(A)$$

is convex since Φ is trace preserving. $\square$

We similarly prove that the entropy gain

$$\varphi(A_1,\ldots,A_k) = S\big(\Phi_1(A_1) + \cdots + \Phi_k(A_k)\big) - \sum_{i=1}^{k} S(A_i)$$

of k positive definite quantities observed through k quantum channels $\Phi_1,\ldots,\Phi_k$ is a convex function in $A_1,\ldots,A_k$.

The first principles

Von Neumann suggested in 1927 the function

$$S(\rho) = -\operatorname{Tr} \rho \log \rho$$

as a measure of quantum entropy based on a gedanken experiment in phenomenological thermodynamics. It enjoys two basic properties:

(i) The entropy of the union of two ensembles is greater than or equal to the average entropy of the component ensembles.

(ii) The incremental information increases when two ensembles are united.

For any convex function $f\colon (0,\infty) \to \mathbf{R}$ the "entropy measure", defined by setting $S_f(\rho) = -\operatorname{Tr} f(\rho)$, has the property that the map

$$\rho \to S_f(\rho) \tag{1}$$

is concave and therefore satisfies the first principle.

Increasing incremental information

The second principle is interpreted as convexity of the map

$$\rho \to S_f(\rho_1) - S_f(\rho) \tag{2}$$

in positive definite operators on a bipartite system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where ρ_1 denotes the partial trace of ρ on $\mathcal{H}_1$.

Lieb and Ruskai (1973) obtained that the von Neumann entropy enjoys this property ($f(t) = t \log t$).

Since partial tracing is a special form of quantum channel one might also interprete the second principle as convexity of the entropy gain

$$\rho \to S_f(\Phi(\rho)) - S(\rho) \tag{2'}$$

over a quantum channel represented by a completely positive trace preserving linear map Φ.

Entropic functions

A quantum channel is represented by a completely positive trace preserving linear map $\Phi\colon B(\mathcal{H}) \to B(\mathcal{K})$.

Definition

Let $f\colon (0,\infty) \to \mathbf{R}$ be a convex function, and let $\mathcal{H}$ be a Hilbert space of dimension n. We say that f is entropic of order n if the function

$$F(\rho) = \operatorname{Tr} f(\rho) - \operatorname{Tr} f(\Phi(\rho))$$

is convex in positive definite operators ρ on $\mathcal{H}$ for any quantum channel Φ.

We say that f is entropic, if it is entropic of all orders.

We associate to an entropic function f a measure of entropy S_f by setting

$$S_f(\rho) = -\operatorname{Tr} f(\rho)$$

for positive definite ρ. Notice that the function $f(t) = t \log t$ is entropic.

Necessary condition for entropicity

Proposition

Let $f : (0, \infty) \to \mathbf{R}$ be entropic of order $2n$. The bivariate function

$$G(\rho, \sigma) = -\operatorname{Tr} f(\rho + \sigma) + \operatorname{Tr} f(\rho) + \operatorname{Tr} f(\sigma)$$

is convex in positive definite operators on a Hilbert space of dimension n.

Proof: We consider the bipartite system

$$\mathcal{H} \otimes l^2(0,1) = \mathcal{H} \oplus \mathcal{H},$$

where $\mathcal{H}$ is of dimension n, and notice that the partial trace is given by

$$\begin{pmatrix} \rho & c \\ c^* & \sigma \end{pmatrix}_1 = \rho + \sigma.$$

Proof continued

Since the partial trace is a quantum channel, we may use the definition of entropicity to obtain that the function

$$A \to -\operatorname{Tr}_1 f(A_1) + \operatorname{Tr}_{12} f(A)$$

is convex in positive definite A on $\mathcal{H} \oplus \mathcal{H}$. We restrict this map to the convex set of diagonal block matrices

$$A = \begin{pmatrix} \rho & 0 \\ 0 & \sigma \end{pmatrix}$$

and obtain that the bivariate function

$$(\rho, \sigma) \to A \to -\operatorname{Tr} f(\rho + \sigma) + \operatorname{Tr} f(\rho) + \operatorname{Tr} f(\sigma)$$

is convex in positive definite operators on $\mathcal{H}$. □

Subentropic functions

Let $f\colon (0,\infty) \to \mathbf{R}$ be a convex function.

Definition

We say that f is subentropic of order n if the function

$$G(\rho,\sigma) = -\operatorname{Tr} f(\rho+\sigma) + \operatorname{Tr} f(\rho) + \operatorname{Tr} f(\sigma)$$

is convex in positive definite operators on a Hilbert space of dimension n.

We say that f is subentropic if it is entropic of all orders.

We notice that an entropic function is subentropic.

We also notice that a convex function satisfying (2) is subentropic.

The Fréchet differential

Let $f\colon (0,\infty) \to \mathbf{R}$ be a continuously differentiable function. The Fréchet differential may be defined by setting

$$df(\rho)h = \lim_{\varepsilon\to 0} \frac{f(\rho+\varepsilon h) - f(\rho)}{\varepsilon}\,,$$

where ρ is positive definite and h is self-adjoint. For operators on a finite dimensional Hilbert space it may be expressed as the Hadamard product

$$df(\rho)h = L_f(\rho) \circ h$$

of h and the Löwner matrix $L_f(\rho)$ in a basis that diagonalises ρ. If L_ρ and R_ρ denote left and right multiplication with operators ρ then

$$\operatorname{Tr} h^* df(\rho)h = \operatorname{Tr} h^*\, k(L_\rho, R_\rho)h,$$

where the bivariate function

$$k(t,s) = \frac{f(t)-f(s)}{t-s} = \int_0^1 f'(\lambda t + (1-\lambda)s)\, d\lambda \qquad t,s>0.$$

Characterisation of subentropic functions

Theorem

A twice continuously differentiable function $f\colon (0,\infty)\to\mathbf{R}$ with strictly positive second derivative is subentropic of order n if and only if

$$df'(\rho+\sigma)^{-1}\geq df'(\rho)^{-1}+df'(\sigma)^{-1}$$

for positive definite operators ρ and σ on a Hilbert space of dimension n.

Proof: The first Fréchet differential of the bivariate function $G(\rho,\sigma)$ in the direction (a,b) is given by

$$\begin{aligned} dG(\rho,\sigma)(a,b) &= d_1G(\rho,\sigma)a + d_2G(\rho,\sigma)b \\ &= -\operatorname{Tr} df(\rho+\sigma)a + \operatorname{Tr} df(\rho)a - \operatorname{Tr} df(\rho+\sigma)b + \operatorname{Tr} df(\sigma)b \\ &= -\operatorname{Tr} f'(\rho+\sigma)(a+b) + \operatorname{Tr} f'(\rho)a + \operatorname{Tr} f'(\sigma)b, \end{aligned}$$

where ρ,σ are positive definite and a,b are self-adjoint.

Proof continued I

The second Fréchet differential in the direction $\big((a,b),(a,b)\big)$ is

$$
\begin{aligned}
d^2G(\rho,\sigma)\big((a,b),(a,b)\big) &= d_1\big(dG(\rho,\sigma)(a,b)\big)a + d_2\big(dG(\rho,\sigma)(a,b)\big)b \\
&= d_1\big(-\operatorname{Tr} f'(\rho+\sigma)(a+b) + \operatorname{Tr} f'(\rho)a + \operatorname{Tr} f'(\sigma)b\big)a \\
&\qquad + d_2\big(-\operatorname{Tr} f'(\rho+\sigma)(a+b) + \operatorname{Tr} f'(\rho)a + \operatorname{Tr} f'(\sigma)b\big)b \\
&= -\operatorname{Tr}(a+b)df'(\rho+\sigma)a + \operatorname{Tr} a df'(\rho)a \\
&\qquad - \operatorname{Tr}(a+b)df'(\rho+\sigma)b + \operatorname{Tr} b df'(\sigma)b \\
&= -\operatorname{Tr}(a+b)df'(\rho+\sigma)(a+b) + \operatorname{Tr} a df'(\rho)a + \operatorname{Tr} b df'(\sigma)b.
\end{aligned}
$$

Since the second Fréchet differential is a symmetric bilinear form we obtain that $G(\rho,\sigma)$ is convex if and only if

$$
\operatorname{Tr}(a+b)^* df'(\rho+\sigma)(a+b) \le \operatorname{Tr} a^* df'(\rho)a + \operatorname{Tr} b^* df'(\sigma)b \qquad (3)
$$

for positive definite ρ,σ and arbitrary a,b.

Proof continued II

The harmonic mean

$$H_2(A, B) = \frac{2}{A^{-1} + B^{-1}}$$

of two positive definite matrices A and B is the maximum of all Hermitian operators C such that

$$\begin{pmatrix} C & C \\ C & C \end{pmatrix} \leq 2 \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Condition (3) may equivalently be written as

$$\left(\begin{pmatrix} a \\ b \end{pmatrix} \middle| \begin{pmatrix} df'(\rho+\sigma) & df'(\rho+\sigma) \\ df'(\rho+\sigma) & df'(\rho+\sigma) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right)_{\mathrm{Tr}}$$

$$\leq \left(\begin{pmatrix} a \\ b \end{pmatrix} \middle| \begin{pmatrix} df'(\rho) & 0 \\ 0 & df'(\sigma) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right)_{\mathrm{Tr}}$$

Proof continued III

and is thus equivalent to the inequality

$$df'(\rho+\sigma) \le H_2\Big(\frac{1}{2}df'(\rho), \frac{1}{2}df'(\sigma)\Big) = \frac{1}{2}H_2\big(df'(\rho), df'(\sigma)\big)$$

for positive definite ρ and σ; where we used that the harmonic mean is positively homogeneous. Since the inverse of the harmonic mean satisfies

$$H_2(A,B)^{-1} = \frac{A^{-1}+B^{-1}}{2}$$

we obtain by taking the inverses of both sides in the above inequality the equivalent inequality

$$df'(\rho+\sigma)^{-1} \ge df'(\rho)^{-1} + df'(\sigma)^{-1}$$

for positive definite ρ, σ. □

Example of subentropic function

The convex function $f(t) = -\log t$ for $t > 0$ is subentropic; meaning that the bivariate operator function

$$(\rho, \sigma) \to \mathrm{Tr}\log(\rho+\sigma) - \mathrm{Tr}\log\rho - \mathrm{Tr}\log\sigma$$

is convex in positive definite operators on a finite dimensional Hilbert space.

Proof: Since $df'(\rho)h = \rho^{-1}h\rho^{-1}$ we realise that

$$df'(\rho)^{-1}h = \rho h \rho.$$

The function $f(t) = -\log t$ is thus subentropic if

$$\mathrm{Tr}\, h^*(\rho+\sigma)h(\rho+\sigma) \geq \mathrm{Tr}\, h^*\rho h\rho + \mathrm{Tr}\, h^*\sigma h\sigma$$

for positive definite ρ, σ and arbitrary h. But this inequality reduces to

$$\mathrm{Tr}\, h^*\rho h\sigma + \mathrm{Tr}\, h^*\sigma h\rho \geq 0$$

which is trivially satisfied.

Matrix entropies

Matrix entropies were introduced by Tropp and Chen as a tool to obtain concentration inequalities for random matrices.

Definition

Let for each natural number n the class Φ_n consist of the functions $f\colon (0,\infty) \to \mathbf{R}$ that are either affine or satisfy

(i) f is convex and twice continuously differentiable.

(ii) The Fréchet differential $df'(\rho)$ is an invertible linear operator and the map $\rho \mapsto df'(\rho)^{-1}$ is concave.

A function $f \in \Phi_n$ is called a matrix entropy of order n.

The class of (representing functions for) matrix entropies Φ_∞ is defined as the intersection

$$\Phi_\infty = \bigcap_{n=1}^{\infty} \Phi_n.$$

Characterisation of matrix entropies

The following conditions are equivalent:

(i) f is the representing function of a matrix entropy.

(ii) The map $(\rho, h) \mapsto \operatorname{Tr} h^* df'(\rho) h$ is convex.

(iii) The function of two variables

$$(\rho, \sigma) \mapsto \operatorname{Tr}(\sigma - \rho)(f'(\sigma) - f'(\rho))$$

is convex in positive definite operators.

(iv) The function of two variables

$$g(t, s) = \frac{s - t}{f'(s) - f'(t)} \qquad t, s > 0$$

is operator concave.

An entropic function is also a matrix entropy

Theorem

A twice continuously differentiable function $f\colon (0,\infty)\to \mathbf{R}$ entropic of order $2n$ is a matrix entropy of order n.

Proof: Let $\mathcal{H}$ be a Hilbert space of dimension $2n$ and assume that f is entropic of order $2n$. We recall that for such a function the entropic gain

$$F(\rho) = -\operatorname{Tr} f(\Phi(\rho)) + \operatorname{Tr} f(\rho), \qquad \rho \text{ positive definite}$$

over a quantum channel Φ is convex. The first Fréchet differential

$$\begin{aligned} dF(\rho)h &= -\operatorname{Tr}_{\mathcal{K}} df\big(\Phi(\rho)\big)\Phi(h) + \operatorname{Tr}_{\mathcal{H}} df(\rho)h \\ &= -\operatorname{Tr}_{\mathcal{K}} f'\big(\Phi(\rho)\big)\Phi(h) + \operatorname{Tr}_{\mathcal{H}} f'(\rho)h. \end{aligned}$$

and the second Fréchet differential is given by

$$d^2F(\rho)(h,h) = -\operatorname{Tr}_{\mathcal{K}} \Phi(h)df'\big(\Phi(\rho)\big)\Phi(h) + \operatorname{Tr}_{\mathcal{H}} hdf'(\rho)h.$$

Proof continued I

The convexity condition for F is therefore equivalent to the inequality

$$\mathrm{Tr}_{\mathcal{K}}\, \Phi(h)^* df'\big(\Phi(\rho)\big)\Phi(h) \le \mathrm{Tr}_{\mathcal{H}}\, h^* df'(\rho)h,$$

where we used that the second Fréchet differential is a symmetric bilinear form. Consider the block matrices

$$U = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad V = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

defined on the direct sum $\mathcal{H} \oplus \mathcal{H}$, where $\mathcal{H}$ now is a Hilbert space of dimension n and put

$$\Phi(X) = UXU^* + VXV^*$$

for $X \in B(\mathcal{H} \oplus \mathcal{H})$. Then Φ is completely positive and satisfies

$$\Phi\begin{pmatrix} \rho & a \\ b & \sigma \end{pmatrix} = \begin{pmatrix} \dfrac{\rho+\sigma}{2} - \dfrac{a+b}{2} & 0 \\ 0 & \dfrac{\rho+\sigma}{2} + \dfrac{a+b}{2} \end{pmatrix}.$$

Proof continued II

We notice that Φ is trace preserving and that

$$\Phi\begin{pmatrix}\rho & 0\\ 0 & \sigma\end{pmatrix}=\frac{\rho+\sigma}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

so Φ is a also unital. Setting

$$h=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\qquad\text{and}\qquad A=\begin{pmatrix}\rho & 0\\ 0 & \sigma\end{pmatrix}$$

we readily obtain from the definition of the Fréchet differential that

$$h^*df'(A)h=\begin{pmatrix}a^*df'(\rho)a & 0\\ 0 & b^*df'(\sigma)b\end{pmatrix}$$

and thus

$$\Phi\bigl(h^*df'(A)h\bigr)=\frac{a^*df'(\rho)a+b^*df'(\sigma)b}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Proof continued III

On the other hand

$$\Phi(h)^* df'(\Phi(A))\Phi(h) = \left[\left(\frac{a+b}{2}\right)^* df'\left(\frac{\rho+\sigma}{2}\right)\left(\frac{a+b}{2}\right)\right]\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

By taking the trace and dividing by 2 we thus obtain

$$\operatorname{Tr}\left(\frac{a+b}{2}\right)^* df'\left(\frac{\rho+\sigma}{2}\right)\left(\frac{a+b}{2}\right) \le \frac{1}{2}\operatorname{Tr} a^* df'(\rho)a + \frac{1}{2}\operatorname{Tr} b^* df'(\sigma)b.$$

The map

$$(\rho, h) \to \operatorname{Tr} h^* df'(\rho)h$$

is thus mid-point convex and by continuity therefore convex.

This implies that f is a matrix entropy of order n. □

The main theorem

Theorem

Let $f : (0, \infty) \to \mathbf{R}$ be a twice continuously differentiable function with strictly positive second derivative. Assume f is entropic of order 2 and normalised such that $f(1) = 0$, $f'(1) = 1$ and $f''(1) = 1$.

Then $f(t) = t \log t$ for $t > 0$.

Proof: Let f be such an entropic function. It follows that f is a matrix entropy of order one. The map

$$\rho \to df'(\rho)^{-1},$$

is thus concave. Since ρ is just a positive number we obtain that the positive function

$$g(t) = \frac{1}{f''(t)} \qquad t > 0$$

is concave.

Proof continued I

Since f is also subentropic of order two we know that

$$df'(\rho+\sigma)^{-1} \geq df'(\rho)^{-1} + df'(\sigma)^{-1}$$

in positive definite operators. In particular,

$$g(t+s) \geq g(t) + g(s) \qquad \text{for positive numbers} \quad t, s > 0.$$

Since g is positive it follows that g is increasing. The existence of the limit

$$g(0) = \lim_{s \to 0} g(s) = 0$$

follows by letting s tend to zero in the above inequality. By replacing s by εs and dividing by ε we thus obtain

$$\frac{g(t+\varepsilon s) - g(t)}{\varepsilon} \geq \frac{g(s) - g(0)}{\varepsilon}$$

Proof continued II

and by letting ε tend to zero this implies the inequality

$$g'_+(t) \geq g'_+(0) \qquad t > 0.$$

This inequality contradicts concavity of the positive function g unless it is affine. Since $g(0) = 0$ there exists thus a constant $b > 0$ such that

$$f''(t)^{-1} = g(t) = bt \qquad t > 0,$$

and since $f''(1) = 1$ we obtain that

$$f''(t) = \frac{1}{t} \qquad t > 0.$$

Since $f'(1) = 1$ we thus obtain $f'(t) = \log t + 1$, and since $f(1) = 1$ finally

$$f(t) = t \log t \qquad t > 0$$

which is the assertion.

A general result for operator monotone functions

Theorem

Let $f : (0, \infty) \to \mathbf{R}$ be a continuously differentiable non-decreasing function. Assume that the Fréchet differential map

$$\rho \to df(\rho)$$

is decreasing in self-adjoint operators ρ on any finite dimensional Hilbert space $\mathcal{H}$. Then f is operator monotone.

Proof: We earlier obtained that

$$\mathrm{Tr}\, h^* df(\rho) h = \mathrm{Tr}\, h^* k(L_\rho, R_\rho) h,$$

where

$$k(t,s) = \frac{f(t) - f(s)}{t - s} \qquad t, s > 0,$$

and L_ρ and R_ρ denote left and right multiplication with ρ.

Proof continued I

We now consider block matrices

$$H = \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad X = \begin{pmatrix} \rho & 0 \\ 0 & \sigma \end{pmatrix}.$$

It is a matter of simple algebra to prove the identities

$$\operatorname{Tr} H^* L_X H = \operatorname{Tr} h^* L_\rho h \quad \text{and} \quad \operatorname{Tr} H^* R_X H = \operatorname{Tr} h^* R_\sigma h.$$

Let $(e_1, \ldots, e_n)$ and $(d_1, \ldots, d_n)$ be orthonormal bases of eigenvectors of ρ and σ such that

$$\rho e_i = \lambda_i e_i \qquad \text{and} \qquad \sigma d_i = \mu_i d_i$$

for $i = 1, \ldots, n$. Setting

$$E_i = \begin{cases} e_i \oplus \underline{0} & i = 1, \ldots, n \\ \underline{0} \oplus d_{i-n} & i = n+1, \ldots, 2n \end{cases}$$

Proof continued II

the orthonormal basis $(E_1, \dots, E_{2n})$ diagonalises X, and since $df(X)H$ is the Hadamard product of the corresponding Löwner matrix $L_f(X)$ and H expressed in this basis we obtain

$$\operatorname{Tr} H^* df(X)H \quad = \sum_{i,j=1}^{n} |(he_i \mid d_j)|^2 \frac{f(\lambda_i) - f(\mu_j)}{\lambda_i - \mu_j} = \operatorname{Tr} h^* k(L_\rho, R_\sigma)h.$$

The assumption thus ensures that the map

$$(\rho, \sigma) \to k(L_\rho, R_\sigma)$$

is separately decreasing. If we choose $\sigma = t$ we obtain that the map

$$\rho \to k(L_\rho, t) = \frac{f(L_\rho) - f(t)}{L_\rho - t}$$

is decreasing.

Proof continued III

However, the left multiplication algebra $\{L_\rho \mid \rho \in B(\mathcal{H})\}$ is isomorphic to $B(\mathcal{H})$. We have thus proved that the operator function

$$\rho \to \frac{f(\rho) - f(t)}{\rho - t}$$

is decreasing in positive definite operators ρ acting on $\mathcal{H}$.

Since the dimension is arbitrary this implies that f is operator concave by Bendat and Sherman's theorem.

Since f is non-decreasing we may by possibly adding a constant assume $f(t) \geq 0$ for $t \geq 1$.

Take positive definite ρ and σ with $\rho < \sigma$. The function

$$\lambda \to \lambda/(1 - \lambda)$$

is increasing in the open interval $(0, 1)$.

Proof continued IV

We thus realise that

$$z = \lambda(1-\lambda)^{-1}(\sigma - \rho) \geq 1 \qquad \text{and thus} \qquad f(z) \geq 0$$

for every λ satisfying $\lambda_0 < \lambda \leq 1$ for some fixed λ_0 sufficiently close to 1. Since by computation

$$\lambda\sigma = \lambda\rho + (1-\lambda)z$$

and f is operator concave, we obtain

$$f(\lambda\sigma) \geq \lambda f(\rho) + (1-\lambda)f(z) \geq \lambda f(\rho)$$

for $\lambda_0 < \lambda < 1$. By letting λ tend to one we then obtain $f(\rho) \leq f(\sigma)$.

By continuity we obtain that f is n-monotone, where n is the dimension of the Hilbert space.

Since n is arbitrary, we finally obtain that f is operator monotone. □

More about subentropic functions

Theorem

Let $f : (0, \infty) \to \mathbf{R}$ be a subentropic function. Then f is operator convex, f' is operator monotone, and the positive function

$$g(t) = \frac{1}{f''(t)} \qquad t > 0$$

is superadditive. In addition, f'' is convex and $f''(t) \to \infty$ for $t \to 0$.

Proof: Since f is subentropic we obtained that

$$df'(\rho + \sigma)^{-1} \geq df'(\rho)^{-1} + df'(\sigma)^{-1} \geq df'(\rho)^{-1}$$

for positive definite ρ and σ. This shows that $\rho \to df'(\rho)^{-1}$ is increasing, and by inversion we obtain that

$$\rho \to df'(\rho)$$

is decreasing.

Proof continued

The convexity of f ensures that f' is non-decreasing. We thus obtain from the preceding theorem that f' is operator monotone and thus on the form

$$f'(t) = \alpha + \beta t + \int_0^\infty \left(\frac{\lambda}{1+\lambda^2} - \frac{1}{t+\lambda} \right) d\nu(\lambda) \qquad t > 0$$

for some non-negative measure ν on the closed half-line $[0, \infty)$ with

$$\int_0^\infty (1+\lambda^2)^{-1} d\nu(\lambda) < \infty$$

and constants α, β with $\beta \geq 0$. From this formula it readily follows that f is operator convex and that f'' is convex.

By following the same line of arguments as earlier, we realise that g is superadditive and that $g(t) \to 0$ for $t \to 0$.

Therefore, $f''(t) \to \infty$ for $t \to 0$. □

Various observations

Let $f\colon (0,\infty) \to \mathbf{R}$ be subentropic. The translation f_ε of f by a positive number ε defined by setting

$$f_\varepsilon(t) = f(\varepsilon + t) \qquad t > 0,$$

is not subentropic. Indeed, the second derivative $f_\varepsilon''(t)$ does not tend to infinity as $t \to 0$.

Example

Let $1 < p \leq 2$. None of the matrix entropies $f(t) = t^p$ are subentropic.

Proof: The positive functions

$$g(t) = \frac{1}{f_p''(t)} = \frac{t^{2-p}}{p(p-1)} \qquad t > 0$$

are strictly concave for $1 < p < 2$ and can therefore not be superadditive. For $p = 2$ the function $g(t) = 1/2$ does not tend to zero for $t \to 0$. □

Literature

The talk is based on the following papers.

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