Conference in honor of Professor Amari

Riemannian interpretation of Wasserstein geometry

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Arnol'd '66: Geometrization of fluid dynamics

Euler's equations for incompressible inviscid fluid, $x \in M = \mathbb{T}^d$:

$$\nabla \cdot u = 0, \quad u = u(t,x) \in \mathbb{R}^d \quad \text{Eulerian velocity}$$

$$\partial_t u + u \cdot \nabla u + \nabla p = 0, \quad p = p(t,x) \in \mathbb{R} \quad \text{pressure}$$

(Formal) Riemannian manifold:

$$\mathcal{M} := \{\Phi \text{ diffeomorphism} \mid \Phi \# dx = dx\} \subset L^2(\mathbb{T}^d, \mathbb{R}^d)$$

For curve $\Phi(t,\cdot)$ in $\mathcal{M}$, consider vector field $u(t,\cdot)$ given by

$\partial_t \Phi(t,\cdot) = u(t,\cdot) \circ \Phi(t,\cdot)$, then

Φ is geodesic in $\mathcal{M}$ $\iff$ u satisfies Euler's equations

Arnol'd '66: an easy calculation

Euler's equations: $\nabla \cdot u = 0$, $\partial_t u + u \cdot \nabla u + \nabla p = 0$.

$\mathcal{M} := \{\Phi \text{ diffeomorphism} \,|\, \Phi\# dx = dx\}$

$= \{\Phi \text{ diffeomorphism} \,|\, \det D\Phi \equiv 1\} \subset L^2(\mathbb{T}^d, \mathbb{R}^d)$.

Φ is geodesic in $\mathcal{M} \iff u$ satisfies Euler's equations,

where $\partial_t \Phi(t) = u(t) \circ \Phi(t)$.

Liouville: $\partial_t \det D\Phi(t) = (\nabla \cdot u)(t) \circ \phi(t) \det D\Phi(t)$

Acceler. Lagrange vs Euler: $\partial_t^2 \Phi(t) = (\partial_t + u \cdot \nabla u)(t) \circ \Phi(t)$.

Arnol'd '66: curvature can get very negative ...

u satisfies Euler's equations $\Longleftrightarrow$ Φ is geodesic in $\mathcal{M}$

where $\partial_t \Phi(t) = u(t) \circ \Phi(t)$.

$\mathcal{M} := \{\Phi \text{ diffeom.} \,|\, \Phi \# dx = dx\}$
$= \{\Phi \text{ diffeom.} \,|\, \det D\Phi \equiv 1\} \subset L^2(\mathbb{T}^d, \mathbb{R}^d)$.

Tangent space in Φ: $T_\Phi \mathcal{M} = \{u \circ \Phi | \nabla \cdot u = 0\} \widehat{=} \{u | \nabla \cdot u = 0\}$

Liouville: $\partial_t \det D\Phi(t) = (\nabla \cdot u)(t) \circ \phi(t) \det D\Phi(t)$

Sectional curvature of $\mathcal{M}$ in plane $u_1 - u_2$

$$R_\Phi(u_1, u_2) = \int A(u_1, u_1) \cdot A(u_2, u_2) - |A(u_1, u_2)|^2 dx$$

where $A(u, u) := \nabla p$ with p solving $\nabla \cdot (u \cdot \nabla u + \nabla p) = 0$

... geodesics diverge, effective unpredictability of Euler

Brenier '91: Projection onto $\mathcal{M}$ ...

$M = (\mathbb{R}^d, d\mu)$ so that
$\mathcal{M} := \{\Phi \text{ diffeomorphism} \,|\, \Phi\# d\mu = d\mu\} \subset L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$.

Given $g \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$ consider $\inf_{\Phi \in \mathcal{M}} \|\Phi - g\|_{L^2_\mu}$.

Existence & uniqueness, solution is of the form

$$g = \nabla\psi \circ \Phi \quad \text{with} \quad \psi \text{ convex}.$$

multi $- d \rightsquigarrow 1 - d$: amounts to monotone rearrangement
nonlinear $\rightsquigarrow$ linear : amounts to Helmholtz projection

... $\rightsquigarrow$ "polar factorization"

Brenier '91: Connection to optimal transportation

Set $\rho := g\#\mu$, then

$$\begin{aligned}
&\inf_{\Phi\in\mathcal{M}} \|\Phi - g\|^2_{L^2_\mu} \\
&= \inf\Big\{ \int_{\mathbb{R}^d} |g-\Phi|^2 d\mu \,\Big|\, \Phi\colon \mathbb{R}^d \to \mathbb{R}^d, \Phi\#\mu = \mu \Big\} \\
&= \inf\Big\{ \int_{\mathbb{R}^d} |\Psi(x) - x|^2 \mu(dx) \,\Big| \\
&\qquad\qquad \Psi\colon \mathbb{R}^d \to \mathbb{R}^d, \Psi\#\mu = \rho \Big\} \quad \text{Monge} \\
&= \inf\Big\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \pi(dxdy) \,\Big|\, \pi \text{ has marginals } \mu, \rho \Big\} \\
&= \sup\Big\{ \int (\tfrac{1}{2}|y|^2 - \varphi(y))\rho(dy) + \int (\tfrac{1}{2}|x|^2 - \psi(x))\mu(dx) \,\Big| \\
&\qquad\qquad \psi,\varphi\colon \mathbb{R}^d \to \mathbb{R}, \varphi(y) + \psi(x) \geq x\cdot y \Big\} \quad \text{Kantorowicz} \\
&= W^2(\rho,\mu) \quad \text{Wasserstein metric}
\end{aligned}$$

McCann '97: displacement convexity

$M = \mathbb{R}^d$. For densities ρ_1 and ρ_0 related via

$\rho_1 = \Psi \# \rho_0$ with $\Psi = \nabla \psi$, ψ convex, see Brenier

consider curve $\rho_s := (s\Psi + (1-s)\mathrm{id})\#\rho_0$, $s \in [0,1]$.

It is a metric geodesic in arc length wrt Wasserstein:

$W(\rho_0, \rho_s) = sW(\rho_0, \rho_1)$ and $W(\rho_s, \rho_1) = (1-s)W(\rho_0, \rho_1)$

Consider functional on densities ρ of form $E(\rho) := \int_{\mathbb{R}^d} U(\rho)dx$.

If U such that $(0, \infty) \ni \lambda \mapsto \lambda^d U(\lambda^{-d})$ convex & decreasing

then E is convex along these geodesics

since A symmetric positive semi-definite $\mapsto$ $(\det A)^{\frac{1}{d}}$ is concave

Barenblatt '52: nonlinear diffusions

Fix $m > 0$. Consider $\rho(t,x) \geq 0$ solution of $\partial_t \rho - \triangle \rho^m = 0$,
wlog $\int \rho dx = 1$.

Admits self-similar solution $\rho_*(t,x) = \frac{1}{t^{d\alpha}} \widehat{\rho}_*(\frac{x}{t^\alpha})$
with $\alpha := \frac{1}{2+(m-1)d}$.

ρ_* describes asymptotic behavior of any solution ρ:

$$t^{d\alpha} \rho(t, t^\alpha \widehat{x}) \overset{t\uparrow\infty}{\to} \widehat{\rho}_*(\widehat{x})$$

Friedman & Kamin '80 based on Caffarelli & Friedman '79

Otto '01: Formal Riemannian structure on space of probability measures

$\mathcal{P}\widehat{=}\{\rho : M \to [0,\infty) | \int_M \rho dx = 1\}$ with metric tensor

$$g_\rho(\delta\rho_1, \delta\rho_2) = \int_M \nabla\varphi_1 \cdot \nabla\varphi_2 \, d\rho$$

where φ_i solves elliptic equation $-\nabla \cdot \rho\nabla\varphi_i = \delta\rho_i$

Connection to Arnol'd for $M = \mathbb{T}^d$:

The map $\Pi\colon L^2(\mathbb{T}^d, \mathbb{R}^d) \to \mathcal{P}$, $\Phi \mapsto \rho = \Phi\# dx$

is Riemannian submersion, $\Pi^{-1}\{dx\} = \mathcal{M}$.

Sectional curvature of $\mathcal{P}$ in plane $\nabla\varphi_1, \nabla\varphi_2$

$$R_\rho(\nabla\varphi_1, \nabla\varphi_2) = \int_{\mathbb{T}^d} |[\nabla\varphi_1, \nabla\varphi_2] - \nabla p|^2 d\rho$$

where p solves $\nabla \cdot \rho([\nabla\varphi_1, \nabla\varphi_2] - \nabla p) = 0$ (O'Neill formula).

Note $R \geq 0$ and $\equiv 0$ if and only if $d = 1$.

Connections to Brenier and McCann

$\mathcal{P} \widehat{=} \{\rho : M \to [0,\infty) | \int \rho dx = 1\}$ endowed with

$$g_\rho(\delta\rho_1, \delta\rho_2) = \int_M \nabla\varphi_1 \cdot \nabla\varphi_2 \, d\rho$$

where φ_i solves $-\nabla \cdot \rho\nabla\varphi_i = \delta\rho_i$

Connection to Brenier for $M = \mathbb{R}^d$:

Wasserstein distance W = induced distance on $\mathcal{P}$

(Benamou-Brenier '00)

Connection to McCann for $M = \mathbb{R}^d$:

displacement convexity = (geodesic) convexity

Nonlinear diffusion = contraction in Wasserstein

Connection to Barenblatt for $M = \mathbb{R}^d$:

nonlinear diffusion $\partial_t \rho - \triangle \rho^m = 0$ is **gradient flow on** $\mathcal{P}$ of $E(\rho) = \int_{\mathbb{R}^d} U(\rho) dx$ with $U(\rho) := \left\{ \begin{array}{cc} \frac{1}{m-1}\rho^m & m \neq 1 \\ \rho \ln \rho & m = 1 \end{array} \right\}$

(Jordan-Kinderlehrer-O.'97)

$m \geq 1 - \frac{1}{d} \iff \lambda \mapsto \lambda^d U(\lambda^{-d})$ convex $\iff$ E **convex on** $\mathcal{P}$

Hence if ρ_i, $i = 1, 2$, solve $\partial_t \rho_i - \triangle \rho_i^m = 0$ then

$$\frac{d}{dt} W^2(\rho_1(t,\cdot), \rho_2(t,\cdot)) \leq 0.$$

In particular $W(t^{d\alpha}\rho(t, t^d \cdot), \hat{\rho}_*) \leq t^{-2\alpha} \int_{\mathbb{R}^d} |x|^2 d\rho(t=0)$

Connections with Ricci curvature

Theorem.

M (compact) d-dim. Riemannian manifold with Ric ≥ 0.

For $m \geq 1 - \frac{1}{d}$ consider $\partial_t \rho_i - \triangle \rho_i^m = 0$, $i = 1, 2$.

Then $$\frac{d}{dt} W^2(\rho_1(t,\cdot), \rho_2(t,\cdot)) \leq 1.$$

O.'01 for $M = \mathbb{R}^d$,

O.&Villani '00 for general M, $m = 1$ (heuristics),

Cordero&McCann&Schmuckenschläger'01,

Sturm&v.Renesse '05 for general M, $m = 1$ (necessity),

O.&Westdickenberg '05

Calculus from differential geometry

Generalize to $\partial_t \rho - \triangle \pi(\rho) = 0$.

Induced distance $\rightsquigarrow$ energy of curves: Given one-parameter family $\{\rho(s,\cdot)\}_{s\in[0,1]}$ of solutions $\partial_t \rho(s,\cdot) - \triangle \pi(\rho(s,\cdot)) = 0$.

Show $\frac{d}{dt}\int_0^1 g_{\rho(s,\cdot)}(\partial_s \rho(s,\cdot), \partial_s \rho(s,\cdot))ds \le 0$.

Infinitesimal version:

Suppose $\partial_t \rho - \triangle \pi(\rho) = 0$ and $\partial_t \delta\rho - \triangle(\pi'(\rho)\delta\rho) = 0$.

Show $\frac{d}{dt} g_\rho(\delta\rho, \delta\rho) \le 0$.

Reduction to single formula

Infinitesimal version:

Suppose $\partial_t \rho - \triangle \pi(\rho) = 0$ and $\partial_t \delta\rho - \triangle(\pi'(\rho)\delta\rho) = 0$.

Show $\frac{d}{dt} g_\rho(\delta\rho, \delta\rho) \leq 0$.

Explicit formula: For $\partial_t \rho - \triangle \pi(\rho) = 0$, $\partial_t \delta\rho - \triangle(\pi'(\rho)\delta\rho) = 0$

and $\delta\rho = -\nabla \cdot (\rho \nabla \varphi)$ have

$$\frac{d}{dt} \int \frac{1}{2} |\nabla \varphi|^2 d\rho$$
$$= -\int (\rho\pi'(\rho) - \pi(\rho))(\triangle \varphi)^2 + \pi(\rho)(|\mathrm{D}^2 \varphi|^2 + \nabla \varphi \cdot \mathrm{Ric} \nabla \varphi) dx$$

Use $(\triangle \varphi)^2 \leq d |\mathrm{D}^2 \varphi|^2$, need $\rho\pi'(\rho) - \pi(\rho) \geq \frac{1}{d} \pi(\rho) \geq 0$

An easy calculation

$$\frac{d}{dt}\int \tfrac{1}{2}|\nabla\varphi|^2 d\rho \quad \text{eliminate } \partial_t\nabla\varphi$$
$$= \int \varphi\partial_t\delta\rho - \tfrac{1}{2}|\nabla\varphi|^2\partial_t\rho dx \quad \text{eliminate } \partial_t\delta\rho,\ \partial_t\rho$$
$$= \int \pi'(\rho)\delta\rho\triangle\varphi - \pi(\rho)\triangle\tfrac{1}{2}|\nabla\varphi|^2 dx \quad \text{eliminate } \delta\rho$$
$$= -\int \rho\pi'(\rho)(\triangle\varphi)^2 + \pi(\rho)(\triangle\tfrac{1}{2}|\nabla\varphi|^2 - \nabla\cdot(\triangle\varphi\nabla\varphi))dx$$

Use Bochner's formula $\triangle\frac{1}{2}|\nabla\varphi|^2 - \nabla\cdot(\triangle\varphi\nabla\varphi)$
$= |D^2\varphi|^2 + \nabla\varphi\cdot\mathrm{Ric}\nabla\varphi - (\triangle\varphi)^2$

Reminiscent of Γ_2-calculus of Bakry-Emery '84

Past – present

Use Wasserstein contraction to give
"synthetic" definition of Ric ≥ 0 on metric spaces M
(Sturm, Lott-Villani, Ambrosio-Gigli-Savaré, ...)

Connections with Ricci flow (McCann-Topping, ...)

Regularity of Brenier map on smooth manifolds M
(Caffarelli+, Trudinger+, Kim, Loeper, Figalli+, ...)

Large deviation principle of underlying particle system
selects the good gradient flow structure
(Dawson&Gärtner, Peletier, Mielke, ...)